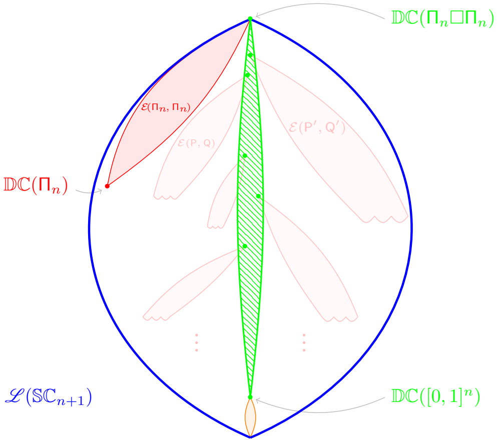
How many generalized permutahedra are there?

Germain Poullot with Georg Loho & Arnau Padrol



24 March 2026, Combinatorial CoWorkspace

arXiv:2510.03177

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Too many?

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Too many?

Yes! Prepare your exponentials!

1. Generalized permutohedra, submodular cone
   - 4 definitions
   - Deformations
2. GP-sums and equality sets
   - $\mathbb{SC}_{n+1} \simeq \mathbb{SC}_n^2$ more or less
   - Drawing $\mathscr{L}(\mathbb{SC}_n)^2$ inside $\mathscr{L}(\mathbb{SC}_{n+1})$
3. Applications (now, we can count stuff)
   - Rays
   - f-vector, total number of faces, non-simplicial faces

Generalized permutohedra, submodular cone

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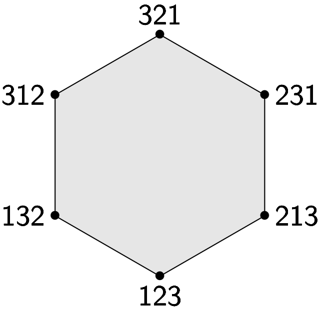
Permutahedron

Example (Permutahedron)

$$\Pi_n = \text{conv}\left\{ \begin{pmatrix} \sigma(1) \\ \vdots \\ \sigma(n) \end{pmatrix} \; ; \; \sigma \text{ permutation of } \{1, \ldots, n\} \right\}$$



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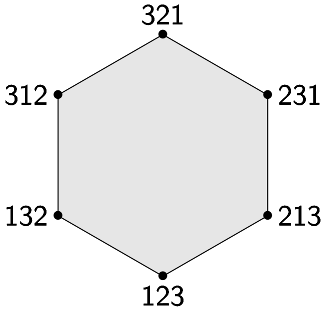
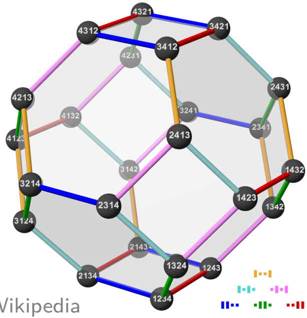
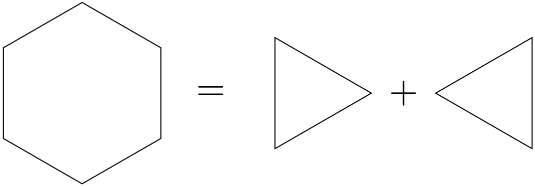




Wikipedia

Definition (Minkowski sum)

P, Q polytopes, $\mathsf{P} + \mathsf{Q} := \{\boldsymbol{p} + \boldsymbol{q} \;\; ; \;\; \boldsymbol{p} \in \mathsf{P}, \; \boldsymbol{q} \in \mathsf{Q}\}$



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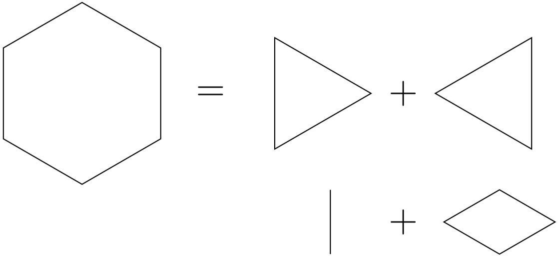

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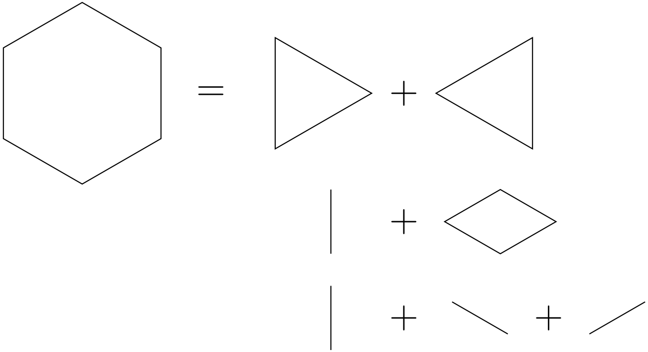

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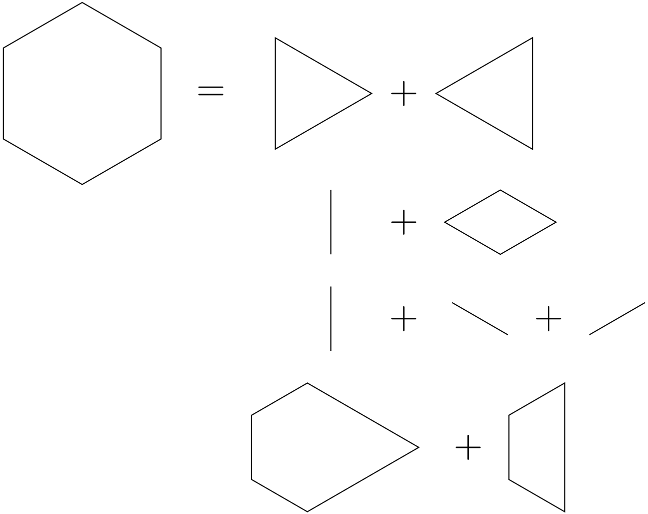

Generalized permutohedra

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Fan coarsening

$\mathcal{N}_{\mathsf{P}}$: Normal fan of polytope P

Coarsening: Choose maximal cones and merge them

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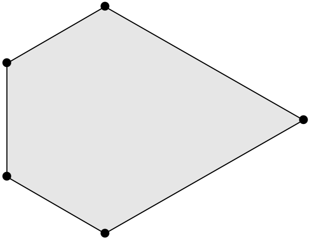
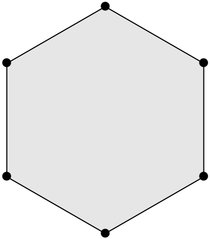
Q is a Minkowski summand of P iff $\mathcal{N}_{\mathsf{Q}}$ coarsens $\mathcal{N}_{\mathsf{P}}$.



summand of



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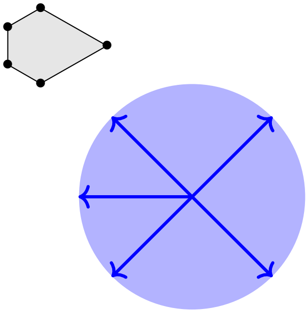

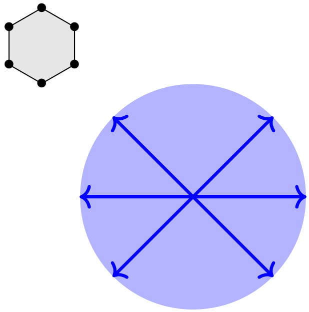

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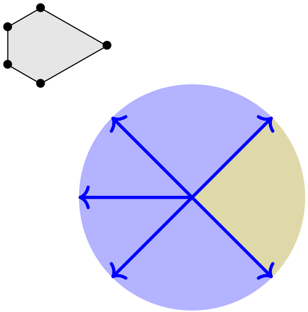
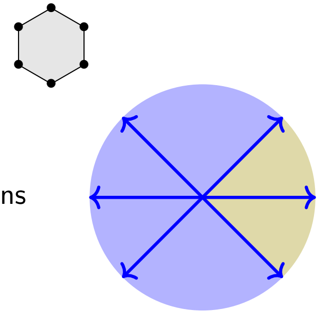


coarsens



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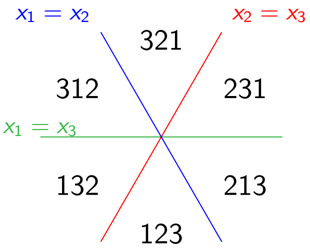

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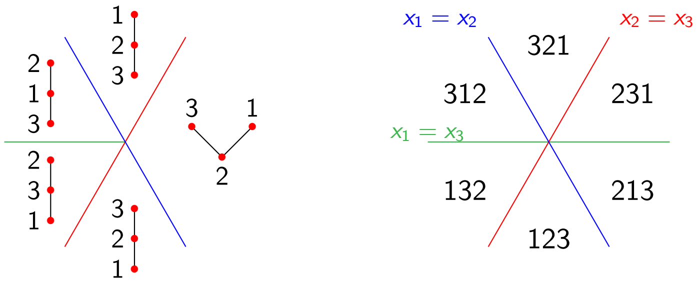

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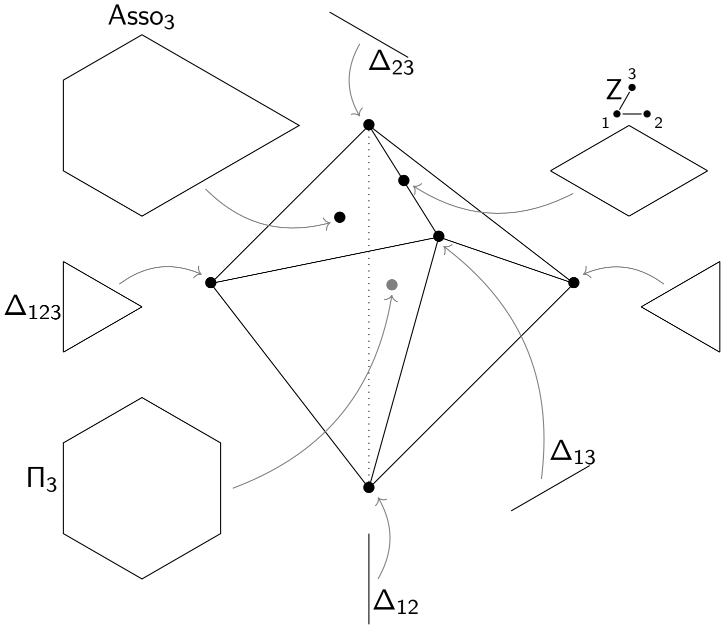
Submodular cone $\mathbb{SC}_n := \{$all generalized n-permutohedra$\}$


Asso3
Δ23
Z
3
1
2
Δ123
Π3
Δ13
Δ12

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$\mathsf{P}, \mathsf{P}' \in \mathbb{SC}_n$

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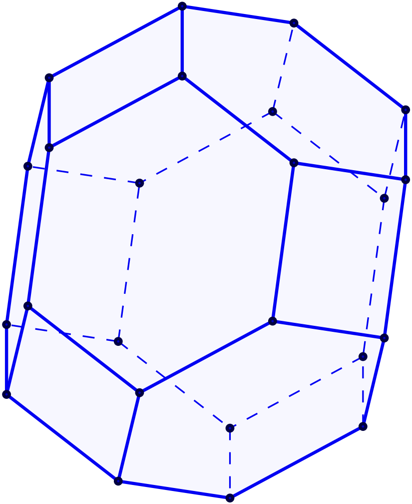
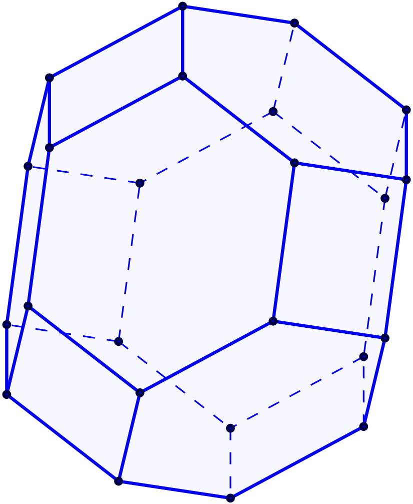
P and P' are *equivalent*, denoted $\mathrm{P} \sim \mathrm{P}'$, iff $\mathrm{P}' \trianglelefteq \mathrm{P}$ and $\mathrm{P} \trianglelefteq \mathrm{P}'$

Deformations of Π_4



Permutahedron Π_4



Sequence of deformations of Π_4

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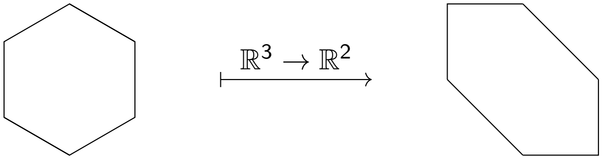
$\dim \mathbb{SC}_n = 2^n - 1, \quad \#_{\text{facets}} \mathbb{SC}_n = \binom{n}{2} 2^{n-2}$,

Edmonds '70: $\#_{\text{rays}} \mathbb{SC}_n =$ please compute it!

Some faces are "known": associahedron, graphical zonotopes (arXiv:2111.12422), nestohedra (arXiv:2109.09200), etc.

GP-sums and equality sets

First, embed $\mathrm{P} \in \mathbb{SC}_{n+1}$ inside $\mathbb{R}^n$: forget last coordinate.
Now, edges directions are $\boldsymbol{e}_i$ and $\boldsymbol{e}_i - \boldsymbol{e}_j$ for $1 \leq i < j \leq n$



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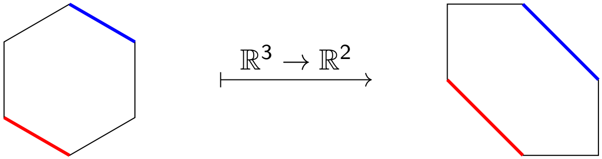

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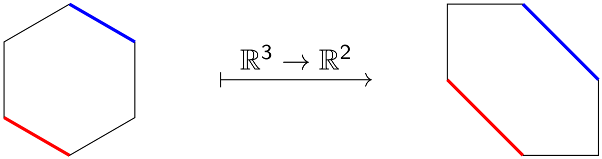

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Thus, $\mathrm{P} \mapsto (\mathrm{P}^+, \mathrm{P}^-)$ *is a bijection:*

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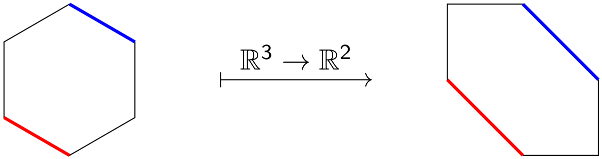

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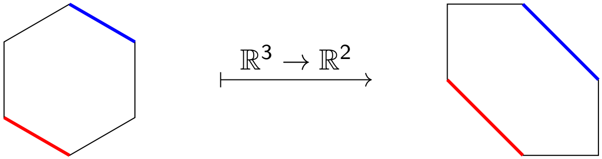

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What's the reciprocal bijection?

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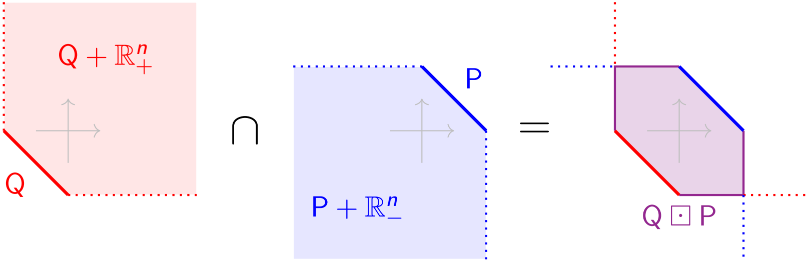

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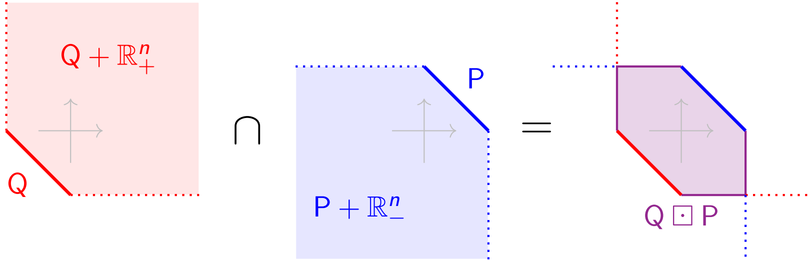
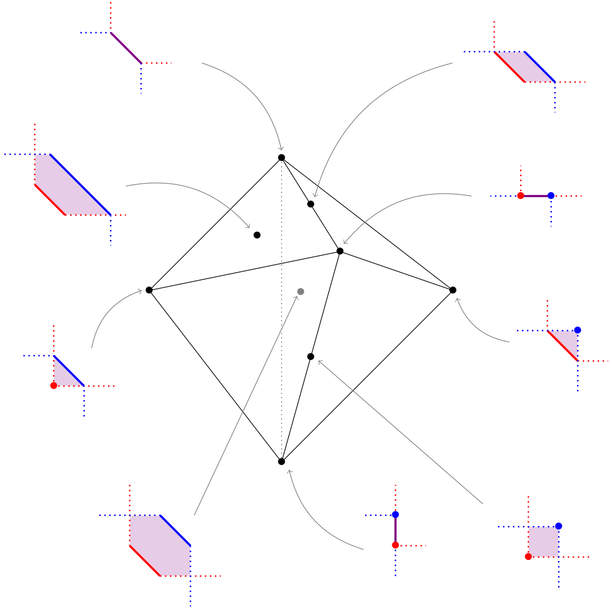


Reciprocal of $\mathsf{P} \mapsto (\mathsf{P}^+, \mathsf{P}^-)$ is the map $(\mathsf{P}, \mathsf{Q}) \mapsto \mathsf{Q} \boxdot \mathsf{P}$

$\mathbb{SC}_3$ drawn with GP-sums



GP-sums

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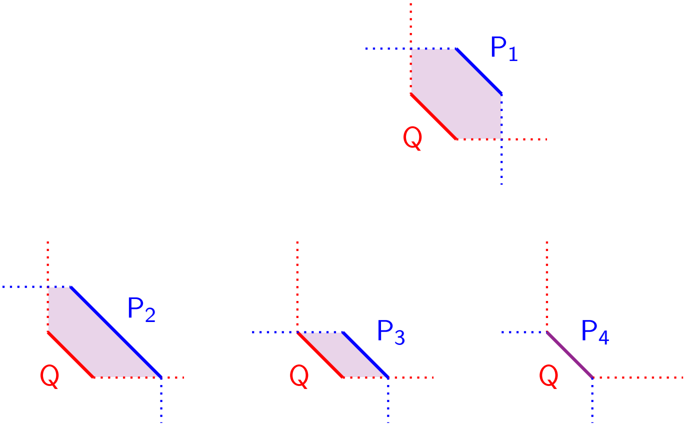

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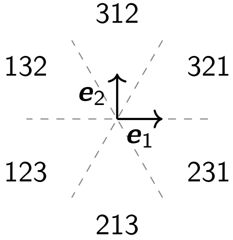
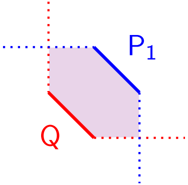
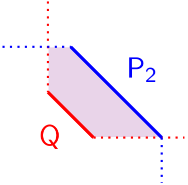
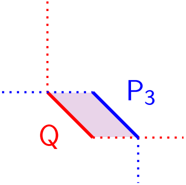
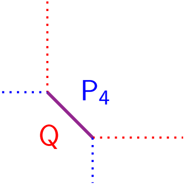
iff for all $i \in [n], \sigma \in S_n$, we have $Q_i^\sigma \leq P_i^\sigma$











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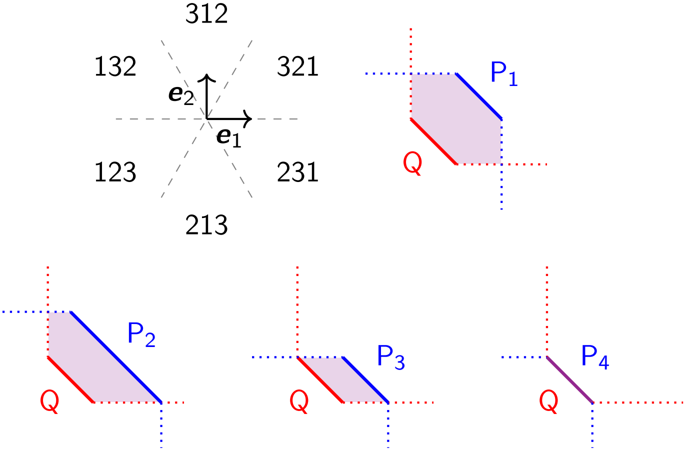

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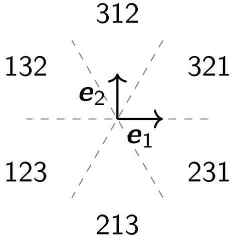

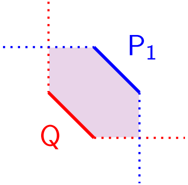

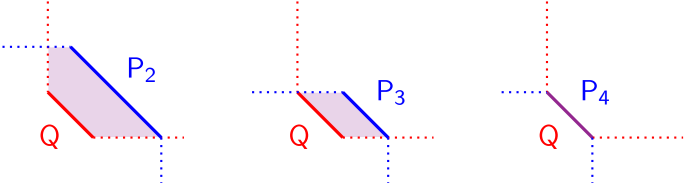

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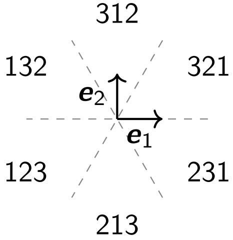
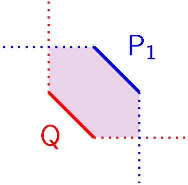
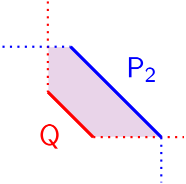
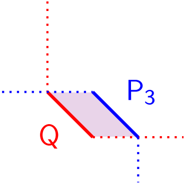






$e_2 : \{213, 231, 321\}$



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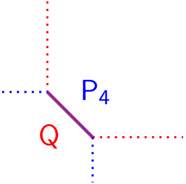

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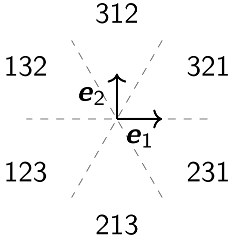
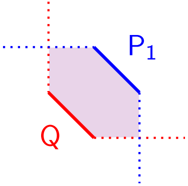
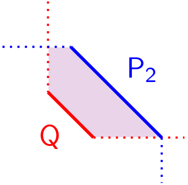




$\varnothing$



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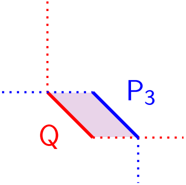

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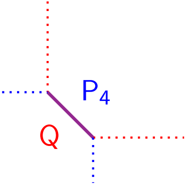

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Poset of valid equality sets

$\mathsf{P}, \mathsf{Q} \in \mathbb{SC}_n$

Definition

Equality set: $\boldsymbol{E}(\mathsf{P}, \mathsf{Q}) := \{(i, \sigma) \in [n] \times \mathcal{S}_n \; ; \; \mathsf{Q}_i^\sigma = \mathsf{P}_i^\sigma\}$
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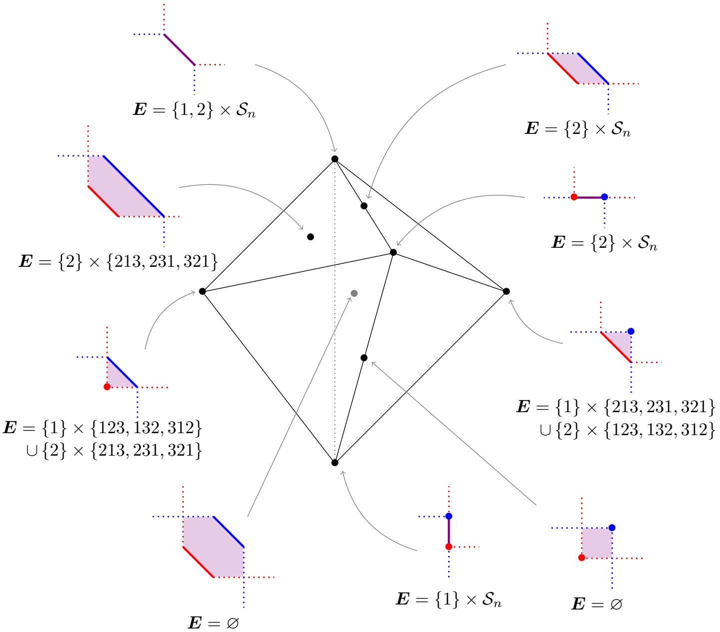
$\mathbb{SC}_3$ with equality sets



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Theorem (Loho–Padrol–P., 25+)

$\mathrm{Q}' \boxdot \mathrm{P}' \sim \mathrm{Q} \boxdot \mathrm{P}$ *iff* $\mathrm{P}' \sim \mathrm{P}$, $\mathrm{Q}' \sim \mathrm{Q}$ *and* $\boldsymbol{E}(\mathrm{P}', \mathrm{Q}') = \boldsymbol{E}(\mathrm{P}, \mathrm{Q})$.

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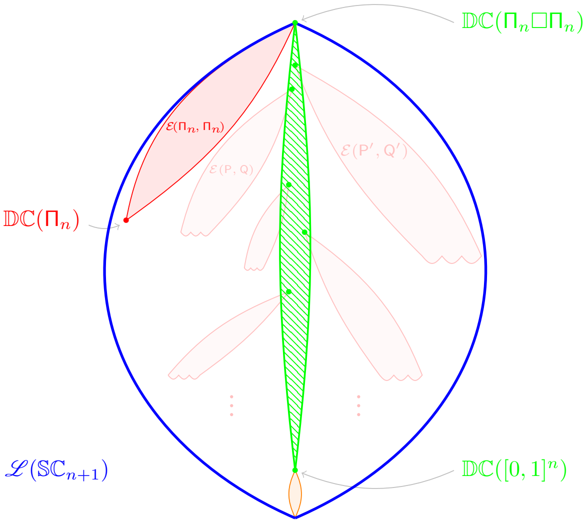
Corollary (Loho–Padrol–P., 25+)

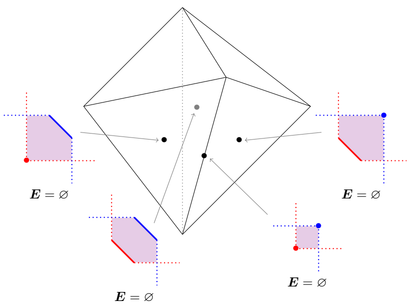
The faces of $\mathbb{SC}_{n+1}$ *are in bijection with triples* $(\mathrm{P},\mathrm{Q},\boldsymbol{E})$ *where* P *and* Q *are taken up to equivalence and* $\boldsymbol{E} \in \mathcal{E}(\mathrm{P},\mathrm{Q})$.

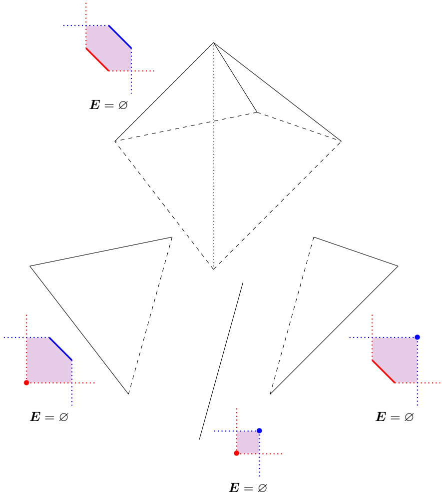
$\mathbb{DC}(\mathsf{P})$: class of $\mathsf{P}' \sim \mathsf{P}$, i.e. faces of $\mathbb{SC}_n$

$$\mathscr{L}(\mathbb{SC}_n)^2 \simeq \{\mathbb{DC}(\mathsf{Q}\square\mathsf{P}) \ ; \ \mathsf{P}, \mathsf{Q} \in \mathbb{SC}_n^2\}$$



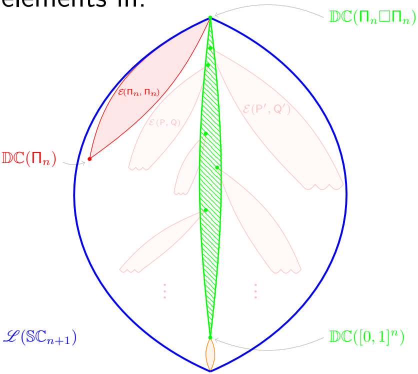
$\mathbb{SC}_3$ partition





Applications (now, we can count stuff)

Rays of $\mathbb{SC}_n$ = indecomposable generalized permutohedra
= bottom-most elements in:



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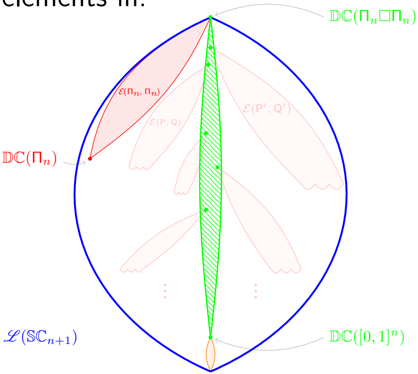

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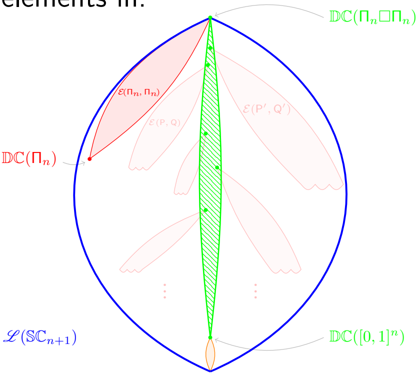

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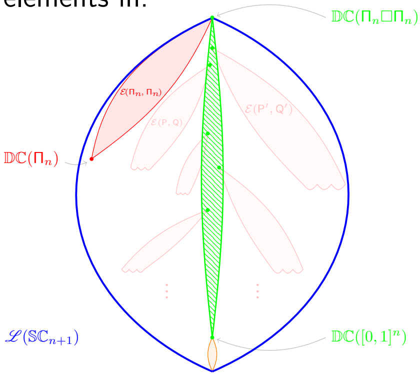

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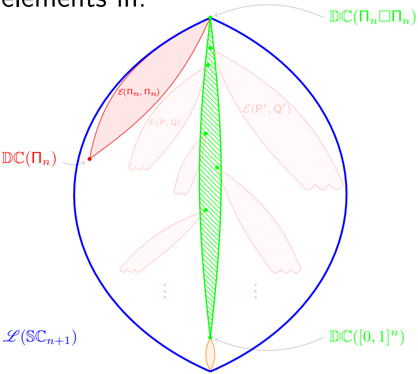
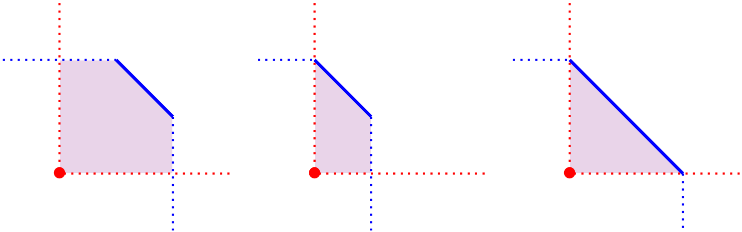


How to get rays of $\mathbb{SC}_{n+1}$?

Start with Q□P, then move P and Q around to increase $\boldsymbol{E}(\mathrm{P}, \mathrm{Q})$, then **hope** for touching the bottom...
To increase hopes, start with $\mathrm{P}, \mathrm{Q} \in$ rays of $\mathbb{SC}_n$!

Rays

P, Q $\in$ rays of $\mathbb{SC}_n$, i.e. deformations are translations & dilations.



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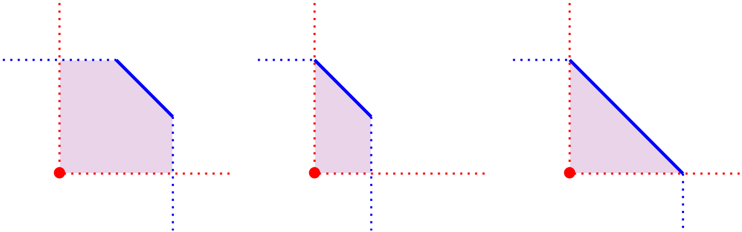

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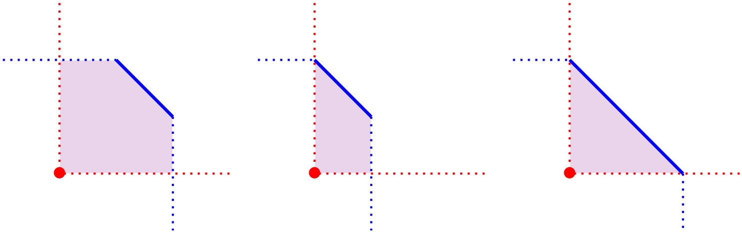

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Problem: Sometimes, you'll need to dilate by 0 to get a ray.

Lemma (Loho–Padrol–P., 25+)

Technical lemma about non-0 dilations that you wouldn't read.

Theorem (Loho–Padrol–P., 25+)

If $P_1, \ldots, P_s$ *are pair-wise fertile rays of* $\mathbb{SC}_n$*, then you can construct explicitly* $R_1, \ldots, R_{s^2}$ *pair-wise fertile rays of* $\mathbb{SC}_{n+1}$.

Number of rays of $\mathbb{SC}_n$

t_n := number of rays of $\mathbb{SC}_n$.

Theorem ((approximately) Nguyen, '86)

The number t_n of rays of $\mathbb{SC}_n$ satisfies

$$2^{2^n/n^{3/2}} \leq t_n$$

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Theorem (Loho–Padrol–P., '25+)

The number t_n of rays of $\mathbb{SC}_n$ satisfies

$$2^{2^{n-2}} \leq t_n$$

N.B.: By Upper Bound Theorem: $t_n \leq n^{2^n}$.

Precisely: $n - 2 \leq \log_2 \log_2 t_n \leq n + \log_2 \log_2 n + 1$

f-vector, and total number of faces

Lemma (Loho–Padrol–P., 25+)

$$\dim \mathbb{DC}(Q \square P) = \dim \mathbb{DC}(P) + \dim \mathbb{DC}(Q) + 1$$

f-vector, and total number of faces

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$F_n(X) := \sum_k f(n,k)X^k$ where $f(n,k) :=$ number k-faces of $\mathbb{SC}_n$

Corollary (Loho–Padrol–P., 25+)

$F_{n+1}(X) \succeq_{\textit{coeff.-wise}} X \cdot F_n(X)^2$

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Hence, $F_{n+1}(X) \succeq_{\textit{coeff.-wise}} F_4^{2^{n-4}}$

And, $\#_{\textit{faces}}\mathbb{SC}_n \geq 22\,108^{2^{n-4}}$

Non-simplicial faces

Lemma (Loho–Padrol–P., 25+)

$\mathbb{DC}(\mathrm{P})$ *not simplicial* $\Rightarrow$ $\mathbb{DC}(\mathrm{Q}\square\mathrm{P})$ *not simplicial for any* $\mathrm{Q} \in \mathbb{SC}_n$.

Non-simplicial faces

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Corollary (Loho–Padrol–P., 25+)

Number of non-simplicial faces of $\mathbb{SC}_n \geq \prod_{p<n} \#_{faces}\mathbb{SC}_p$

In particular, $\mathbb{SC}_n$ is "very far" from being simplicial, so we should be "very far" for the Upper Bound Theorem equality case.

What's more?

Furthermore:

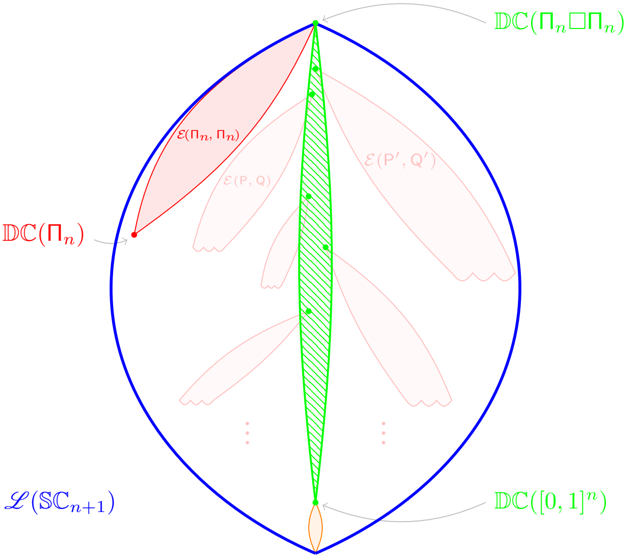
• $\mathscr{L}(\mathbb{SC}_n)^2 \simeq \{\mathbb{DC}(\mathrm{Q}\square\mathrm{P}); \mathrm{P}, \mathrm{Q} \in \mathbb{SC}_n^2\}$ is the face figure of $[0,1]^n$.

• $\mathcal{E}(\mathrm{P}, \mathrm{P})$ is iso. to the interval of $\mathscr{L}(\mathbb{SC}_{n+1})$ between P and $\mathrm{P}\square\mathrm{P}$.
In particular, $\mathcal{E}(\Pi_n, \Pi_n)$ is a face figure.

• In general, $\mathcal{E}(\mathrm{P}, \mathrm{Q})$ is a co-graded, co-atomistic, join-semi-lattice with a geometric interpretation.

• Notion of independence polytope Ind_P for all generalized permutohedra, similar to matroid polytopes, with:
Ind_P indecomposable iff P indecomposable, and
$\mathrm{Ind}_{\mathrm{P}+\mathrm{Q}} = \mathrm{Ind}_\mathrm{P} + \mathrm{Ind}_\mathrm{Q}$

Thank you!



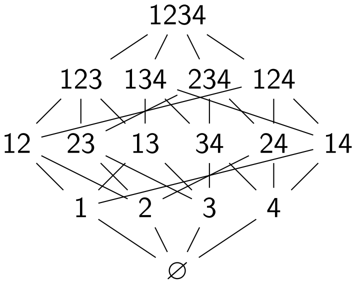
$\mathbb{DC}(\Pi_n \square \Pi_n)$
$\mathcal{E}(\Pi_n, \Pi_n)$
$\mathcal{E}(P', Q')$
$\mathcal{E}(P, Q)$
$\mathbb{DC}(\Pi_n)$
$\mathscr{L}(\mathbb{SC}_{n+1})$
$\mathbb{DC}([0,1]^n)$

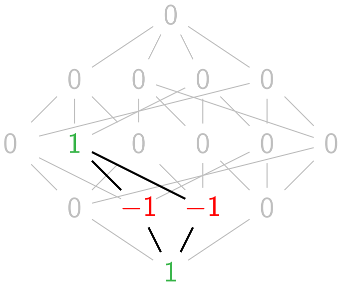
Tools: submodular dependancies

Notations: $Sx = S \cup \{x\}$, $(\boldsymbol{f}_X)_{X \subseteq [n]}$ canonical basis of $\mathbb{R}^{2^{[n]}}$

Definition

Submodular vector $\boldsymbol{n}(S,u,v) = \boldsymbol{f}_{Suv} - \boldsymbol{f}_{Su} - \boldsymbol{f}_{Sv} + \boldsymbol{f}_S$
for $u, v \in S \subseteq [n]$





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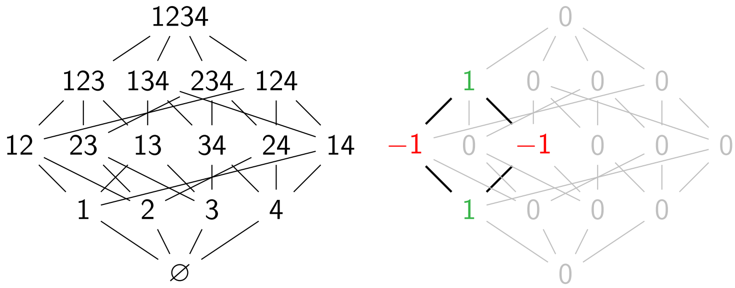


Tools: submodular dependancies

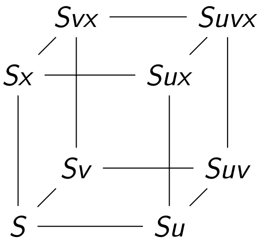
Lemma (Submodular normal)

$\boldsymbol{n}(S, u, v)$ *are the facet's normals of* $\mathbb{DC}(\Pi_n)$

Lemma (Cubic relation)

$u, v, x \notin S \subseteq [n]$

$$\boldsymbol{n}(Suvx, u, v) + \boldsymbol{n}(Sux, u, x) = \boldsymbol{n}(Suv, u, v) + \boldsymbol{n}(Suvx, u, x)$$



NB: Cubic relations generates all relations of submodular vectors

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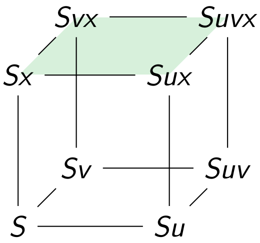

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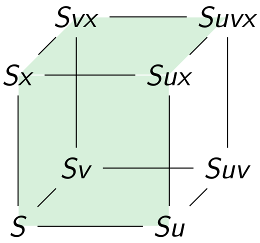

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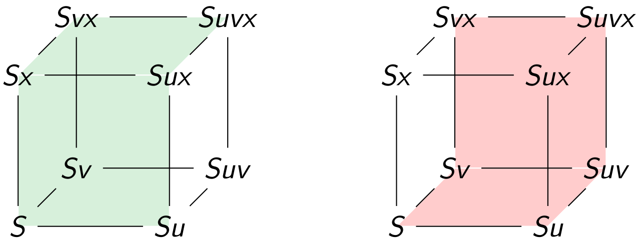

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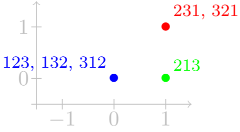
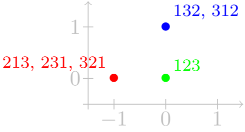
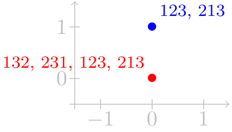
(P, Q) is *fertile* iff there exists $i \in [n]$ such that for all $\tau \in S_n$, either $P_i^\tau \neq \min_{\sigma \in S_n} P_i^\sigma$ or $Q_i^\tau \neq \max_{\sigma \in S_n} Q_i^\sigma$



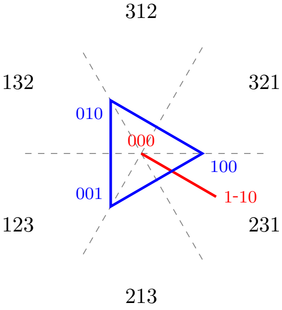
$i = 1$



$i = 2$



$i = 3$



Multiple choice questions
(sometimes several answer possible)

A sum of 2 co-planar triangles can have	3 edges	4 edges	5 edges	6 edges
$\mathcal{N}_{P+Q} =$ ___ of $\mathcal{N}_P, \mathcal{N}_Q$	union	intersection	common refinement	product
Edge directions of GP $\in$	$\{e_i\}_i$	$\{e_i + e_j\}_{i,j}$	$\{e_i - e_j\}_{i,j}$	$\{e_i \pm e_j\}_{i,j}$
Z_G has no hexagonal face iff G is	a tree	K_3-free	K_4-free	complete
___ form a basis of $\mathrm{Vect}(\mathbb{SC}_n)$	$\{\Delta_X\}_{X\subseteq[n]}$	nestohedra	shard polytopes	matroid polytopes
$\dim \mathbb{SC}_3 =$	3	4	5	11
Group of symmetries of $\mathbb{SC}_n$:	Free_n	$\mathcal{S}_n$	$\mathcal{S}_n \times \mathbb{Z}_2$	$\mathcal{S}_n \times \mathbb{Z}_n$
# 3-dim indecomposable GP:	3	5	7	37

1. Show that $\mathrm{Vert}(\mathrm{P}+\mathrm{Q}) \subseteq \{\boldsymbol{u}+\boldsymbol{v} \,;\, \boldsymbol{u} \in \mathrm{Vert}(\mathrm{P}),\ \boldsymbol{v} \in \mathrm{Vert}(\mathrm{Q})\}$. Find an example with strict inclusion. Prove that $\mathcal{N}_{\mathrm{P}+\mathrm{Q}}$ is the common refinement of $\mathcal{N}_{\mathrm{P}}$ and $\mathcal{N}_{\mathrm{Q}}$.

2. Find all the ways to write the 2-dimensional permutahedron as a Minkowski sum of indecomposable polytopes.

3. Prove that $\boldsymbol{h}_{\mathrm{P}+\mathrm{Q}} = \boldsymbol{h}_{\mathrm{P}} + \boldsymbol{h}_{\mathrm{Q}}$ and $\boldsymbol{\ell}_{\mathrm{P}+\mathrm{Q}} = \boldsymbol{\ell}_{\mathrm{P}} + \boldsymbol{\ell}_{\mathrm{Q}}$. What are $\boldsymbol{h}_{\mathrm{P}+\boldsymbol{t}}$ and $\boldsymbol{\ell}_{\mathrm{P}+\boldsymbol{t}}$?

4. Prove that a triangle is Minkowski indecomposable. Deduce that all simplicial polytopes are Minkowski indecompsable.

5. Take vectors $\boldsymbol{s}, \boldsymbol{s}', \boldsymbol{r}_1, \ldots, \boldsymbol{r}_{d-1}$ such that $\mathrm{C} = \mathrm{cone}(\boldsymbol{s}, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_{d-1})$ and $\mathrm{C}' = \mathrm{cone}(\boldsymbol{s}', \boldsymbol{r}_1, \ldots, \boldsymbol{r}_{d-1})$ are simplicial cones which intersect on their proper face $\mathrm{C} \cap \mathrm{C}' = \mathrm{cone}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_{d-1})$. Show that there exist $\alpha_{\boldsymbol{s}}, \alpha_{\boldsymbol{s}'} > 0$ and $\alpha_{\boldsymbol{r}_i} \in \mathbb{R}$ such that:

$$\alpha_{\boldsymbol{s}} \boldsymbol{s} + \alpha_{\boldsymbol{s}'} \boldsymbol{s}' + \sum_i \alpha_{\boldsymbol{r}_i} \boldsymbol{r}_i = \boldsymbol{0}$$

6. Compute (and draw) the deformation cone of a parallelogram (use the edge-lengths point of view). What about a parallelopiped? Show that there is a **unique** way to write a parallelopiped P as a sum of Minkowski indecomposable polytopes.

7. Show that $\mathbb{SC}_n$ has $\binom{n}{2} 2^{n-2}$ facets. Give bounds on its number of rays.

8. Show that graphical zonotopes are generalized permutahedra. Show that nestohedra are generalized permutahedra. Show that matroid polytopes are generalized permutahedra. Which are indecomposable?

9. If G is triangle-free, then $\mathbb{DC}(\mathrm{Z}_G)$ is simplicial (see lecture). Each face of $\mathbb{DC}(\mathrm{Z}_G)$ is associated to a polytope: which polytope?

10. The *weighted graphical zonotope* of a graph $G = (V, E)$ and weight (on edges) $\omega : E \to \mathbb{R}_+^*$ is $\mathrm{Z}_{G,\omega} := \sum_{(i,j)\in E} \omega(i,j)[\boldsymbol{e}_i, \boldsymbol{e}_j]$. Show that: $\{$zonotopes$\} \cap \{$generalized permutahedra$\} = \{$weighted graphical zonotopes$\}$. Show that the cone of weighted graphical zonotopes is a section (i.e. intersection with a linear sub-space) of the submodular cone, such that the rays of this section are rays of $\mathbb{SC}_n$. What is the dimension of this section?

11. Prove the cubic relations hold. Show that cubic relations generates all linear relations between submodular vectors.